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Quantum state of an injected TROPO above threshold: purity, Glauber function and photon number distribution

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Abstract. In this paper, we investigate several properties of the full signal-idler-pump mode quantum state generated by a triply resonant non-degenerate Optical Parametric Oscillator operating above threshold, with an injected wave on the signal and idler modes in order to lock the phase diffusion process. We determine and discuss the spectral purity of this state, which turns out not to be always equal to 1 even though the three interacting modes have been taken into account at the quantum level. We have seen that the purity is essentially dependent on the weak intensity of the injected light and on an asymmetry of the synchronization. We then derive the expression of its total three-mode Glauber P-function, and calculate the joint signal-idler photon number probability distribution and investigate their dependence on the injection.

PACS. 42.50.Dv Nonclassical states of the electromagnetic field, including entangled photon states; quantum state engineering and measurements – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps

1 Introduction

Three-wave nonlinear interaction in a χ^2 medium is one of the main model systems of quantum optics. The problem is simplified when one inserts an optical cavity around the non-linear medium, because the resonances of the cavity permit to restrict the analysis to the intracavity resonant modes only. The system is called in this case a "TROPO" (Triply Resonant Optical Parametric Oscillator). In addition, in the bellow threshold configuration, one has generally the possibility to consider the pump as a coherent one with a fixed classical amplitude and thus to restrict the analysis to a two-mode problem (idler and signal). It has been shown that such a system produces a squeezed vacuum state below the oscillation threshold when the two quantum modes are degenerate [1], and twin, quantum intensity correlated beams above the threshold when they are non-degenerate [2]. Both predictions have been confirmed over the last two decades by many experiments [3, 4]. It has been also shown that this system has many other interesting quantum properties, especially in the non-degenerate case: phase anti-correlations [5], and EPR entanglement below and above threshold [6]. This last feature has been recently confirmed experimentally [7,8].

In the above-threshold regime, where the three fields have intensities which are of the same order of magnitude, the assumption that the intracavity pump field is classical is not valid. One is faced with a true three-quantum-mode problem. In such a system, not only a conversion of the pump wave into the signal and idler waves, but a mutual conversions of all three modes take place. It was theoretically shown [5] that under this conditions the pump wave turns out to be non-classical too. It turns out to be significantly squeezed in some parts of the parameter space, and this was confirmed experimentally [9]. More recently it has been shown that the pump beam is quantum-correlated with the sum of the signal and idler fields, and that there is actually a strong three-partite EPR entanglement between the three interacting fields [10]. This shows that the TROPO, which has been one of the most studied systems by quantum opticians for decades, can still provide us with good surprises.

Moreover, in the recent years, the attention of the quantum optics community has gradually shifted from the determination of squeezed variances and quantum correlations of various observables of the system to more subtle characterizations and manipulations of its quantum state. There is for instance a strong development of studies concerning states produced by conditional measurements performed on continuous variable optical systems, such as

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photon deletion techniques or selection of instantaneous values of the photocurrent fluctuations. For such studies, one needs to know the full quantum state of the system under consideration, and not only the second order moment of the observed quantities.

Whereas it is simple to define a quantum state of a confined optical system, such as a cavity mode or a single light pulse, it turns out to be very difficult to rigorously define the quantum state of a c.w. optical beam propagating from the generating optical device to the detectors, even though such a concept is often used in the community from an intuitive point of view. One difficulty that one encounters is that the state of the system crucially depends on the exact measurement that one wants to perform on it. For example, quantum properties are present in the system when measured using long integration times, and the system is perfectly classical for short integration times.

The purpose of this paper is to bring some insight to this important issue by considering as an example the "toy-model" of the light generated by a TROPO, and to present various ways of characterizing its full quantum state. We will consider here only the above threshold case, where all the generated fields have significant mean values, so that the linearization method for treating the fluctuations [11,12] is undoubtedly valid except very close to the oscillation threshold. As is well-known, as far as the signal-idler phase difference is concerned, a phase diffusion phenomenon takes places, giving rise to diverging phase difference fluctuations at very long times [13]. To eliminate this complication, we will assume here that the TROPO is injected on the signal and/or idler modes, which has the effect of synchronizing the two fields and locking the phase diffusion effect. In this case, the validity of the method is unquestionable, as well as the Gaussian character of all the output fields. A similar approach was developed for the degenerate OPO by Drummond et al. [14], in this case phase diffusion is no relevant.

As a starting point we will determine the full 6×6 covariance matrix for the spectral components of the quantum fluctuations of the three output modes. This enables us to calculate the spectral purity of the TROPO quantum state, and to discuss in which respect the system can be described or not by a three-field state vector. We then derive the expression of the intracavity stationary Glauber function for the whole system. Its interest lies in the fact that the quasi-probability function contains in a condensed form all the quantum properties. The way it is written, and the symmetries that it reveals, are a guide to determine which are the combinations of the different modes which have the best quantum properties. We use it to determine the P-function and purity of the outfield state in a simple particular case. We finally derive the joint signal-idler probability distribution, a useful quantity to predict, using a quantum state-reduction approach, the result of conditional measurements performed on the system [15].

The article is organized as follows. In Section 2, we give the framework of the model that we use to describe the injected TROPO above threshold. We then derive the expression of the Fourier components of the quantum fluctuations of the three interacting modes. In the next section, we determine and discuss the purity of the system from the expression of the covariance matrix. We then derive the Glauber stationary P function for the intracavity fields. We use it to derive the expression of the Glauber stationary P function for the output fields in the simple case where the exposure time of the detection is small. Details of the derivation in the asymmetrical injection case are given in the appendix of the paper.

2 Physical model and general equations

2.1 The master equation

A non-degenerate optical parametric oscillator consists of a non-linear $\chi^{(2)}$ medium inserted in a high-Q cavity. This medium ensures the down-conversion process $\omega_p \to \omega_i + \omega_s$ with exchange of a pump photon with twin signal and idler photons. As is well-known this interaction can be described in the exact phase matching case by the effective interaction Hamiltonian [13,16]:

$$\hat{V} = i\hbar g \left(\hat{a}_p \hat{a}_i^\dagger \hat{a}_s^\dagger - h.c. \right) \quad (1)$$

$$\left[\hat{a}_l, \hat{a}_l^\dagger \right] = 1, \quad l = p, i, s.$$

In this approach, the three interacting intra-cavity modes are quantized. The master equation in the interaction picture for three-mode field density matrix $\hat{\rho}$ is:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar} \left[\hat{V}, \hat{\rho} \right] - \hat{R}\hat{\rho} + \sum_{m=p,i,s} \hat{D}_m \hat{\rho}. \quad (2)$$

The operator $\hat{R}$ describes the damping of the intracavity quantum oscillators and its action on the density matrix is determined by the following equality in the case of exact resonance between the pump, signal, idler fields and three cavity modes:

$$\hat{R}\rho = \sum_{m=p,i,s} \frac{\kappa_m}{2} \left(\hat{a}_m^\dagger \hat{a}_m \hat{\rho} + \hat{\rho} \hat{a}_m^\dagger \hat{a}_m - 2\hat{a}_m \rho \hat{a}_m^\dagger \right) \quad (3)$$

where κ_m is the energy damping rate of the m-mode. The operators $\hat{D}_m$ ensure an excitation of each of the actual modes by a quasi-classical or coherent state of amplitude $\sqrt{N_m^{in}}$:

$$\hat{D}_m \, \hat{\rho} = \frac{\kappa_m}{2} \sqrt{N_m^{in}} \left[\hat{a}_m^\dagger - \hat{a}_m, \hat{\rho} \right]. \quad (4)$$

The external quasi-classical fields with a real amplitudes $\sqrt{N_m^{in}}$ are in resonance with the m-waves.

The excitation of the idler and signal waves by the external coherent waves is used to depress the phase diffusion in the system. For this aim, it would be enough to put $\sqrt{N_i^{in}} \equiv \sqrt{N^{in}} \neq 0$ and $\sqrt{N_s^{in}} = 0$. However the

asymmetry arising here complicates both the mathematical and physical situations. Much simpler solutions take place as $\sqrt{N_i^{in}} = \sqrt{N_s^{in}} \equiv \sqrt{N^{in}}$. In this article we shall consider both models focusing in the body of the article on the symmetrical case and giving all wished formulas for the asymmetrical excitation in Appendix A.

We can derive from equation (2) the corresponding evolution equation in the Glauber diagonal representation for the three interacting modes. The Glauber function P is introduced by the integral relation:

$$\hat{\rho}(t) = \iiint P(\alpha_p, \alpha_i, \alpha_s, t) \, |\alpha_p, \alpha_i, \alpha_s\rangle$$
$$\times \, \langle \alpha_p, \alpha_i, \alpha_s | d^2\alpha_p \, d^2\alpha_i \, d^2\alpha_s. \quad (5)$$

With symmetrical phase locking, the master equation reads:

$$\frac{\partial P(\alpha_p, \alpha_i, \alpha_s, t)}{\partial t} = \sum_{m=i,s} \frac{\kappa_m}{2} \frac{\partial}{\partial \alpha_m} (\alpha_m - \sqrt{N^{in}}) P$$
$$+ \frac{\kappa_p}{2} \frac{\partial}{\partial \alpha_p} (\alpha_p - \sqrt{N_p^{in}}) P + g \left(\alpha_i \alpha_s \frac{\partial P}{\partial \alpha_p} - \alpha_p \alpha_s^* \frac{\partial P}{\partial \alpha_i} \right.$$
$$\left. - \alpha_p \alpha_i^* \frac{\partial P}{\partial \alpha_s} \right) + g \alpha_p \frac{\partial^2 P}{\partial \alpha_i \partial \alpha_s} + c.c. \quad (6)$$

The Glauber representation is often un-practical, as, when non-classical effects are present, the Glauber functions can be expressed only in terms of distributions, difficult to handle mathematically. Sudarshan has suggested some way for writing these distributions [17,18] in the form of the momentum series, however the master equation remains often impossible to express simply. For instance, for the simplest model of the sub-Poissonian laser [19], the master equation contains derivatives of all orders with respect to complex amplitudes, which means that all momenta are connected with each other in one system of equations of infinite order. In our problem, the situation happens to be very favorable as the master equation has a simple and well defined expression. Thus, even if the positive P representation is usually more suitable as it has no divergence problems, in our case the P representation is enough and simpler as defined on a twice as small phase space.

2.2 Classical equations for the OPO operation

As is well-known, to get the semi-classical evolution equations for the mean amplitudes, we can use equation (6), neglecting the second derivatives with respect to the complex amplitude. Then one can get:

$$\dot{\alpha}_p = -\frac{\kappa_p}{2} \left(\alpha_p - \sqrt{N_p^{in}} \right) - g \alpha_i \alpha_s \quad (7)$$

$$\dot{\alpha}_i = -\frac{\kappa_i}{2} \left(\alpha_i - \sqrt{N^{in}} \right) + g \alpha_p \alpha_s^* \quad (8)$$

$$\dot{\alpha}_s = -\frac{\kappa_s}{2} \left(\alpha_s - \sqrt{N^{in}} \right) + g \alpha_p \alpha_i^*. \quad (9)$$

In the following we will only consider the case of equal losses for the signal and idler cavity losses:

$$\kappa_i = \kappa_s \equiv \kappa. \quad (10)$$

Then it is not difficult to obtain the stationary solutions, which can be written in the form:

$$\alpha_p = \sqrt{N_p}, \qquad \alpha_i = \alpha_s = \sqrt{N}, \quad (11)$$

where for the values N and N_p the following equalities take place:

$$g\sqrt{N_p} = \frac{\kappa}{2}(1 - \mu), \qquad \frac{gN}{\sqrt{N_p}} = \frac{\kappa_p}{2}(\mu_p - 1),$$
$$\kappa(1 - \mu)N = \kappa_p(\mu_p - 1)N_p. \quad (12)$$

These equations depend on two dimensionless parameters μ_p and μ. μ_p, the pump parameter, is defined as

$$\mu_p = \sqrt{\frac{N_p^{in}}{N_{th}}} \quad (13)$$

where $N_{th} = \kappa^2(1 - \mu)/(4g^2)$. $\mu_p > 1$ corresponds to the above threshold regime. μ, the injection parameter, represents which fraction of the total signal and/or idler fields is injected:

$$\mu = \sqrt{\frac{N^{in}}{N}}. \quad (14)$$

We will restrict our analysis to $\mu \ll 1$. There are two reasons for this. First, we are going to apply the limit of small photon number fluctuation that is impossible in the bistability area. Second, for a strong injected field, the Poissonian statistics of the external field would be imposed on the intracavity mode and its quantum properties would be destroyed.

2.3 Limit of small amplitude and phase fluctuations

We make now two important assumptions. First of all, because we consider the above threshold situation inside a high-Q cavity, the limit of the small photon number fluctuations can be used in our treatment. If we present the complex amplitudes via amplitudes and phases

$$\alpha_m = \sqrt{u_m} \, e^{i\varphi_m}, \qquad m = p, i, s, \quad (15)$$

then we can require

$$u_m = N_m + \varepsilon_m, \qquad \varepsilon_m \ll N_m. \quad (16)$$

Secondly, because of injection, one can find well defined values for the steady state phases, namely $\varphi_m = 0$. We will assume that in the system the phase fluctuations around these steady state values are small:

$$\varphi_m \ll 1. \quad (17)$$

Taking into account this limit of small amplitude and phase fluctuations, one finds that it is possible to factorize the Glauber distribution in the form:

$$P(\alpha_p, \alpha_i, \alpha_s, t) = P(\varepsilon_p, \varepsilon_+, t)P(\varepsilon_-, t)P(\varphi_p, \varphi_+, t)P(\varphi_-, t)$$
(18)

with decoupled equations for the different factors:

$$\frac{\partial P(\varepsilon_p, \varepsilon_+, t)}{\partial t} = \left(\frac{1}{2} \frac{\partial}{\partial \varepsilon_p}(\kappa_p \varepsilon_p + \kappa(1-\mu)\varepsilon_+) \right.$$
$$+ \frac{\partial}{\partial \varepsilon_+} \left(\frac{\mu\kappa}{2}\varepsilon_+ - \kappa_p(\mu_p - 1)\varepsilon_p \right)$$
$$\left. + \kappa N(1-\mu)\frac{\partial^2}{\partial \varepsilon_+^2} \right) P(\varepsilon_p, \varepsilon_+, t), \quad (19)$$

$$\frac{\partial P(\varepsilon_-, t)}{\partial t} = \left(\kappa(1-\mu/2)\frac{\partial}{\partial \varepsilon_-}\varepsilon_- \right.$$
$$\left. - \kappa N(1-\mu)\frac{\partial^2}{\partial \varepsilon_-^2} \right) P(\varepsilon_-, t), \quad (20)$$

$$\frac{\partial P(\varphi_p, \varphi_+, t)}{\partial t} = \left(\frac{\kappa_p}{2}\frac{\partial}{\partial \varphi_p}(\varphi_p + (\mu_p - 1)\varphi_+) \right.$$
$$+ \frac{\partial}{\partial \varphi_+}(\kappa(1-\mu/2)\varphi_+ - \kappa(1-\mu)\varphi_p)$$
$$\left. - \frac{\kappa}{4N}(1-\mu)\frac{\partial^2}{\partial \varphi_+^2} \right) P(\varphi_p, \varphi_+, t), \quad (21)$$

$$\frac{\partial P(\varphi_-, t)}{\partial t} = \left(\frac{\mu\kappa}{2}\frac{\partial}{\partial \varphi_-}\varphi_- + \frac{\kappa}{4N}(1-\mu)\frac{\partial^2}{\partial \varphi_-^2} \right)$$
$$\times P(\varphi_-, t). \quad (22)$$

Here

$$\varepsilon_\pm = \varepsilon_i \pm \varepsilon_s, \quad \varphi_\pm = \varphi_i \pm \varphi_s. \quad (23)$$

One sees that the photon and phase fluctuations turn out to be statistically independent at exact triple resonance. Furthermore, the last equation shows that the injected fields lead to the locking of the differential phase and to the suppression of the phase diffusion phenomenon present in the degenerate OPO above threshold. However the synchronizing fields influence the noise properties of the system. It is important to understand whether the phase locking and the presence of significant noise reduction and quantum correlation are compatible with each other or not. This is what we will see in the following.

2.4 Intracavity spectral densities

In order to analyze the time dependent correlation function, let us derive the Langevin equations for the three interacting fields (19)–(22). They are easily written ac-

cording to well-known rules and have the following form:

$$\dot{\varepsilon}_p = -\kappa_p/2\,\varepsilon_p - \kappa/2(1-\mu)\,\varepsilon_+, \quad (24)$$
$$\dot{\varepsilon}_+ = -\kappa\mu/2\,\varepsilon_+ + \kappa_p(\mu_p - 1)\,\varepsilon_p + f_+(t), \quad (25)$$
$$\dot{\varepsilon}_- = -\kappa(1-\mu/2)\,\varepsilon_- + f_-(t), \quad (26)$$
$$\dot{\varphi}_p = -\kappa_p/2\,\varphi_p - \kappa_p/2(\mu_p - 1)\,\varphi_+, \quad (27)$$
$$\dot{\varphi}_+ = -\kappa(1-\mu/2)\,\varphi_+ + \kappa(1-\mu)\,\varphi_p + g_+(t), \quad (28)$$
$$\dot{\varphi}_- = -\kappa\mu/2\,\varphi_- + g_-(t), \quad (29)$$

where the stochastic sources are determined by the pair correlation functions

$$\langle f_+(t)f_+(t')\rangle = 2\kappa N(1-\mu)\,\delta(t-t'), \quad (30)$$
$$\langle f_-(t)f_-(t')\rangle = -2\kappa N(1-\mu)\,\delta(t-t'), \quad (31)$$
$$\langle g_+(t)g_+(t')\rangle = -\kappa(1-\mu)/(2N)\,\delta(t-t'), \quad (32)$$
$$\langle g_-(t)g_-(t')\rangle = \kappa(1-\mu)/(2N)\,\delta(t-t'). \quad (33)$$

The best way to solve these equations is to rewrite them in the Fourier domain and solve the simple algebraic system of equations for the spectral components. One obtains

$$(\varepsilon_+^2)_\omega = 2N \quad (34)$$
$$\times \frac{\kappa(\kappa_p^2 + 4\omega^2)(1-\mu)}{[2\omega^2 - \kappa_p\kappa[\mu/2 + (\mu_p - 1)(1-\mu)]]^2 + \omega^2(\kappa_p + \kappa\mu)^2},$$

$$(\varepsilon_p^2)_\omega = 2N_p \quad (35)$$
$$\times \frac{\kappa^2\kappa_p(1-\mu)^2(\mu_p - 1)}{[2\omega^2 - \kappa_p\kappa[\mu/2 + (\mu_p - 1)(1-\mu)]]^2 + \omega^2(\kappa_p + \kappa\mu)^2},$$

$$(\varepsilon_p\varepsilon_+)_\omega = -2N \quad (36)$$
$$\times \frac{\kappa^2\kappa_p(1-\mu)}{[2\omega^2 - \kappa_p\kappa[\mu/2 + (\mu_p - 1)(1-\mu)]]^2 + \omega^2(\kappa_p + \kappa\mu)^2},$$

$$(\varepsilon_-^2)_\omega = -2N\,\frac{\kappa(1-\mu)}{\kappa^2(1-\mu/2)^2 + \omega^2}, \quad (37)$$

and

$$(\varphi_p^2)_\omega = -\frac{1}{2N_p} \quad (38)$$
$$\times \frac{\kappa^2\kappa_p(\mu_p - 1)(1-\mu)^2}{[2\omega^2 - \kappa\kappa_p[\mu/2 + \mu_p(1-\mu)]]^2 + \omega^2[\kappa_p + 2\kappa(1-\mu/2)]^2},$$

$$(\varphi_+^2)_\omega = -\frac{1}{2N} \quad (39)$$
$$\times \frac{\kappa(\kappa_p^2 + 4\omega^2)(1-\mu)}{[2\omega^2 - \kappa\kappa_p[\mu/2 + \mu_p(1-\mu)]]^2 + \omega^2[\kappa_p + 2\kappa(1-\mu/2)]^2},$$

$$(\varphi_p\varphi_+)_\omega = \frac{1}{2N} \quad (40)$$
$$\times \frac{\kappa\kappa_p^2(\mu_p - 1)(1-\mu)}{[2\omega^2 - \kappa\kappa_p[\mu/2 + \mu_p(1-\mu)]]^2 + \omega^2[\kappa_p + 2\kappa(1-\mu/2)]^2},$$

$$(\varphi_-^2)_\omega = \frac{1}{2N}\,\frac{\kappa(1-\mu)}{(\kappa\mu/2)^2 + \omega^2}. \quad (41)$$

In these expressions, the spectral density $(\varepsilon_m^2)_\omega$ and $(\varphi_m^2)_\omega$ are defined as a factor in front of the delta-functions in the

correlation functions

$$\langle \varepsilon_m(\omega)\, \varepsilon_m(\omega')\rangle = (\varepsilon_m^2)_\omega\, \delta(\omega + \omega'), \qquad m = p, \pm,$$

$$\langle \varphi_m(\omega)\, \varphi_m(\omega')\rangle = (\varphi_m^2)_\omega\, \delta(\omega + \omega'), \tag{42}$$

where

$$\varepsilon_m(\omega) = \frac{1}{\sqrt{2\pi}} \int \varepsilon_m(t)\, e^{-i\omega t} dt,$$

$$\varepsilon_m(t) = \frac{1}{\sqrt{2\pi}} \int \varepsilon_m(\omega)\, e^{i\omega t} d\omega. \tag{43}$$

The mutual correlations $(\varepsilon_p \varepsilon_+)_\omega$ and $(\varphi_p \varphi_+)_\omega$ are defined exactly in the same way:

$$\langle \varepsilon_+(\omega)\, \varepsilon_p(\omega')\rangle = (\varepsilon_+ \varepsilon_p)_\omega\, \delta(\omega + \omega'),$$

$$\langle \varphi_+(\omega)\, \varphi_p(\omega')\rangle = (\varphi_+ \varphi_p)_\omega\, \delta(\omega + \omega'). \tag{44}$$

3 Spectral purity of the OPO quantum state

3.1 Output field variances and correlations

In the previous section, we have considered the intracavity spectral densities. Now we want to determine the corresponding quantities for the output beams. We consider the optimum case where only one mirror of the cavity is not perfectly reflecting and transmits the light onto the detectors. Inside the cavity the normalized amplitude was defined by the photon operators $\hat{a}_m(t)$ $(m = p, i, s)$ that obey the commutation relations

$$\left[\hat{a}_m(t), \hat{a}_m^\dagger(t)\right] = \delta_{mn}. \tag{45}$$

Let us call $\hat{A}_m(t)$ the corresponding operator for the output beams. They obey the commutation relations:

$$\left[\hat{A}_m(t), \hat{A}_n(t')^\dagger\right] = \delta_{mn}\delta(t - t'),$$

$$\left[\hat{A}_m(t), \hat{A}_n(t')\right] = 0 \tag{46}$$

and are related to the intracavity ones by the input-output relations on the coupling mirror:

$$\hat{A}_m(t) = \sqrt{\kappa_m}\, \hat{a}_m(t) - \left(C_m + \hat{A}_{m,vac}(t)\right),$$

$$m = p, i, s. \tag{47}$$

We have taken into account the fact that the reflection coefficient of the output mirror is about one. C_p is the complex normalized amplitude of the pump wave in resonance with the p-mode. This amplitude is related to the quantity N_p^{in} introduced earlier by relation $C_p = \sqrt{\kappa_p N_p^{in}/2}$. The amplitudes C_s and C_i are the classical injected fields in resonance with the idler and signal modes: $C_i = C_s = \sqrt{\kappa N^{in}}/2$.

$\hat{A}_{m,vac}(t)$ are the input vacuum fluctuations, with commutation relations

$$\left[\hat{A}_{m,vac}(t), \hat{A}_{n,vac}(t')^\dagger\right] = \delta_{mn}\delta(t - t'),$$

$$\left[\hat{A}_{m,vac}(t), \hat{A}_{n,vac}(t')\right] = 0. \tag{48}$$

We can rewrite (47) for the fluctuations $\delta\hat{A}_m = \hat{A}_m - \langle\hat{A}_m\rangle$ and $\delta\hat{a}_m = \hat{a}_m - \langle\hat{a}_m\rangle$, in the Fourier domain:

$$\delta\hat{A}_m(\omega) = \sqrt{\kappa_m}\, \delta\hat{a}_m(\omega) - \hat{A}_{m,vac}(\omega). \tag{49}$$

Let us divide the frequency scale into small equal intervals of size Δ. Then for each discrete frequency ω_l one can write:

$$\delta\hat{A}_m^l = \frac{1}{\sqrt{\Delta}} \int_{\omega_l - \Delta/2}^{\omega_l + \Delta/2} \delta\hat{A}_m(\omega)\, d\omega. \tag{50}$$

It is easy to check that the algebra of these operators is determined by the commutation relations:

$$\left[\delta\hat{A}_m^l, (\hat{A}_m^k)^\dagger\right] = \delta_{lk}. \tag{51}$$

Thus the operators $\delta\hat{A}_m^l$, $(\hat{A}_m^l)^\dagger$ can be thought of as being the annihilation and creation operators for the photons in the frequency band around ω_l. Then for each mode (frequency) one can introduce the corresponding quadrature components via the standard relations

$$\delta\hat{X}_m^l = \frac{1}{2}\left((\delta\hat{A}_m^l)^\dagger + \delta\hat{A}_m^l\right),$$

$$\delta\hat{Y}_m^l = \frac{i}{2}\left((\delta\hat{A}_m^l)^\dagger - \delta\hat{A}_m^l\right). \tag{52}$$

Taking into account (49) the variances of the output quadratures can then be expressed in terms of the intracavity ones:

$$\langle\delta\hat{X}_m^2\rangle_l = \frac{1}{4} + \frac{\kappa_m}{4N_m} \frac{1}{\Delta} \int_{\omega_l - \Delta/2}^{\omega_l + \Delta/2} (\varepsilon_m^2)_\omega\, d\omega,$$

$$\langle\delta\hat{Y}_m^2\rangle_l = \frac{1}{4} + \kappa_m N_m \frac{1}{\Delta} \int_{\omega_l - \Delta/2}^{\omega_l + \Delta/2} (\varphi_m^2)_\omega\, d\omega. \tag{53}$$

Going to the continuous frequencies by means of $\Delta \to 0$, one can obtain the wished expression giving the observed variances and correlation in function of the intracavity ones:

$$4(\delta\hat{X}_m^2)_\omega = 1 + \kappa_m/N_m\, (\varepsilon_m^2)_\omega,$$

$$4(\delta\hat{Y}_m^2)_\omega = 1 + 4\kappa_m N_m\, (\varphi_m^2)_\omega. \tag{54}$$

$$4(\delta\hat{X}_m \delta\hat{X}_n)_\omega = \sqrt{\kappa_m/N_m}\sqrt{\kappa_n/N_n}\, (\varepsilon_m\varepsilon_n)_\omega,$$

$$4(\delta\hat{Y}_m \delta\hat{Y}_n)_\omega = 4\sqrt{\kappa_m N_m}\sqrt{\kappa_n N_n}\, (\varphi_m\varphi_n)_\omega, \tag{55}$$

$$m, n = p, i, s.$$

3.2 Spectral purity

Let us now introduce the three-mode spectral covariance matrix of the OPO. In the present case of exact cavity resonance for the three modes, the amplitude and phase

fluctuations are independent and this matrix turns out to be quasi-diagonal:

$$\mathcal{M}_\omega = \begin{pmatrix} \mathcal{M}_\epsilon & 0 \\ 0 & \mathcal{M}_\phi \end{pmatrix}, \qquad (56)$$

where $||\mathcal{M}_\epsilon||$ are amplitude and $||\mathcal{M}_\phi||$ are phase 3×3 matrices given by:

$$\mathcal{M}_\epsilon = \begin{pmatrix} 4(\delta X_p^2)_\omega & 4(\delta X_p \delta X_i)_\omega & 4(\delta X_p \delta X_s)_\omega \\ 4(\delta X_i \delta X_p)_\omega & 4(\delta X_i^2)_\omega & 4(\delta X_i \delta X_s)_\omega \\ 4(\delta X_s \delta X_p)_\omega & 4(\delta X_s \delta X_i)_\omega & 4(\delta X_s^2)_\omega \end{pmatrix}, \quad (57)$$

and analogically $||\mathcal{M}_\phi||$ with help of the variances $4(\delta Y_m \delta Y_n)_\omega$. The spectral variances are determined as a factor in front of the delta-function in the correlation relation:

$$\langle \delta X_m(\omega) \delta X_n(\omega') \rangle = (\delta X_m \delta X_n)_\omega \, \delta(\omega + \omega'),$$
$$\langle \delta Y_m(\omega) \delta Y_n(\omega') \rangle = (\delta Y_m \delta Y_n)_\omega \, \delta(\omega + \omega'). \qquad (58)$$

A widely used way of describing the properties of a quantum system at a given noise frequency ω is to assign to it a *quantum state*, that we will call "single noise frequency state", described by the density matrix $\hat{\rho}_\omega$, that can be experimentally characterized for example by quantum tomography. Let us introduce the *spectral purity*, given by

$$\Pi_\omega = \mathrm{Tr} \hat{\rho}_\omega^2. \qquad (59)$$

This state is a pure state when $\Pi_\omega = 1$, or a statistical mixture when $\Pi_\omega < 1$. In the present case of small fluctuations and input Gaussian states, all quantum fluctuations have Gaussian statistics. Then the purity is equal to

$$\Pi_\omega = \frac{1}{\sqrt{\det \mathcal{M}_\omega}}. \qquad (60)$$

3.3 Spectral purity for a symmetrical injection

In this simple case, we will use the basis of sum and difference modes $m, n = p, \pm$ to calculate the determinant and therefore the purity. The determinant of the covariance matrix can then be written as the product:

$$\det \mathcal{M}_\omega = \mathcal{N}_- \mathcal{D}_X \mathcal{D}_Y, \qquad (61)$$

where

$$\mathcal{N}_- = 2\langle \delta \hat{X}_-^2 \rangle_\omega 2\langle \delta \hat{Y}_-^2 \rangle_\omega,$$
$$\mathcal{D}_X = 2\langle \delta \hat{X}_+^2 \rangle_\omega 4\langle \delta \hat{X}_p^2 \rangle_\omega - 4\langle \delta \hat{X}_+ \delta \hat{X}_p \rangle_\omega^2,$$
$$\mathcal{D}_Y = 2\langle \delta \hat{Y}_+^2 \rangle_\omega 4\langle \delta \hat{Y}_p^2 \rangle_\omega - 4\langle \delta \hat{Y}_+ \delta \hat{Y}_p \rangle_\omega^2. \qquad (62)$$

One can see that $\mathcal{N}_-$ is related to the uncertainty relation for the differential variances. $\mathcal{D}_{X,Y}$ is the determinant of the 2×2 covariance matrix for the correlated sum and pump amplitude and phase variances.

In this subsection we are going to consider, as an example and for the simplicity of the calculation, the case

$\kappa(\mu_p - 1), \kappa \ll \kappa_p$. According to the previous subsection the differential variances are derived in the form

$$2\langle \delta \hat{X}_-^2 \rangle_\omega = 1 + \frac{\kappa}{2N}(\varepsilon_-^2)_\omega$$
$$= 1 - \frac{\kappa^2(1-\mu)}{\kappa^2(1-\mu/2)^2 + \omega^2}, \qquad (63)$$

$$2\langle \delta \hat{Y}_-^2 \rangle_\omega = 1 + 2\kappa N(\varphi_-^2)_\omega = 1 + \frac{\kappa^2(1-\mu)}{(\kappa\mu/2)^2 + \omega^2}. \qquad (64)$$

From these expressions, one retrieves first a striking but well-known result, characteristic of the non-degenerate OPO above threshold: $\mathcal{N}_- = 1$ for any values of the parameters such as the frequency, μ_p, and μ.

In order to get the other determinants $\mathcal{D}_{X,Y}$ we need to derive the corresponding variances in the explicit form:

$$2\langle \delta \hat{X}_+^2 \rangle_\omega = 1 + \frac{\kappa}{2N}(\varepsilon_+^2)_\omega$$
$$= 1 + \frac{\kappa^2(1-\mu)}{\kappa^2[\mu/2 + (1-\mu)(\mu_p - 1)]^2 + \omega^2}, \qquad (65)$$

$$2\langle \delta \hat{Y}_+^2 \rangle_\omega = 1 + 2\kappa N(\varphi_+^2)_\omega$$
$$= 1 - \frac{\kappa^2(1-\mu)}{\kappa^2[\mu/2 + (1-\mu)\mu_p]^2 + \omega^2}, \qquad (66)$$

$$4\langle \delta \hat{X}_p^2 \rangle_\omega = 1 + \frac{\kappa_p}{N_p}(\varepsilon_p^2)_\omega$$
$$= 1 + \frac{2\kappa^2(\mu_p - 1)(1-\mu)^2}{\kappa^2[\mu/2 + (1-\mu)(\mu_p - 1)]^2 + \omega^2}, \qquad (67)$$

$$4\langle \delta \hat{Y}_p^2 \rangle_\omega = 1 + 4\kappa_p N_p(\varphi_p^2)_\omega$$
$$= 1 - \frac{2\kappa^2(1-\mu)^2(\mu_p - 1)}{\kappa^2[\mu/2 + (1-\mu)\mu_p]^2 + \omega^2}, \qquad (68)$$

$$2\langle \delta \hat{X}_p \delta \hat{X}_+ \rangle_\omega = \sqrt{\frac{\kappa_p}{2N_p}} \sqrt{\frac{\kappa}{2N}} (\varepsilon_p \varepsilon_+)_\omega$$
$$= -\frac{\kappa^2 \sqrt{(1-\mu)(\mu_p - 1)}}{\kappa^2[\mu/2 + (\mu_p - 1)(1-\mu)]^2 + \omega^2}, \qquad (69)$$

$$2\langle \delta \hat{Y}_p \delta \hat{Y}_+ \rangle_\omega = \sqrt{2\kappa_p N_p} \sqrt{2\kappa N} (\varphi_p \varphi_+)_\omega$$
$$= \frac{\kappa^2 \sqrt{(\mu_p - 1)(1-\mu)^3}}{\kappa^2[\mu/2 + \mu_p(1-\mu)]^2 + \omega^2}. \qquad (70)$$

Let us remark that, in contrast to $\mathcal{N}_-$, the quantities

$$\mathcal{N}_+ = 2\langle \delta \hat{X}_+^2 \rangle_\omega \, 2\langle \delta \hat{Y}_+^2 \rangle_\omega,$$
$$\mathcal{N}_p = 4\langle \delta \hat{X}_p^2 \rangle_\omega \, 4\langle \delta \hat{Y}_p^2 \rangle_\omega \qquad (71)$$

are non minimum, especially near zero frequencies when the pump is not strongly above threshold. For example, when $\mu_p - 1 \gg \mu$ and $\omega = 0$

$$\mathcal{N}_+ = \mathcal{N}_p = \frac{(\mu_p - 1)^2 + 1}{\mu_p^2} \frac{\mu_p + 1}{\mu_p - 1} \qquad (72)$$

$\mathcal{N}_+$ and $\mathcal{N}_p$ take the minimum value compatible with the Heisenberg inequality when the OPO is strongly above threshold, but take very large values close to threshold.
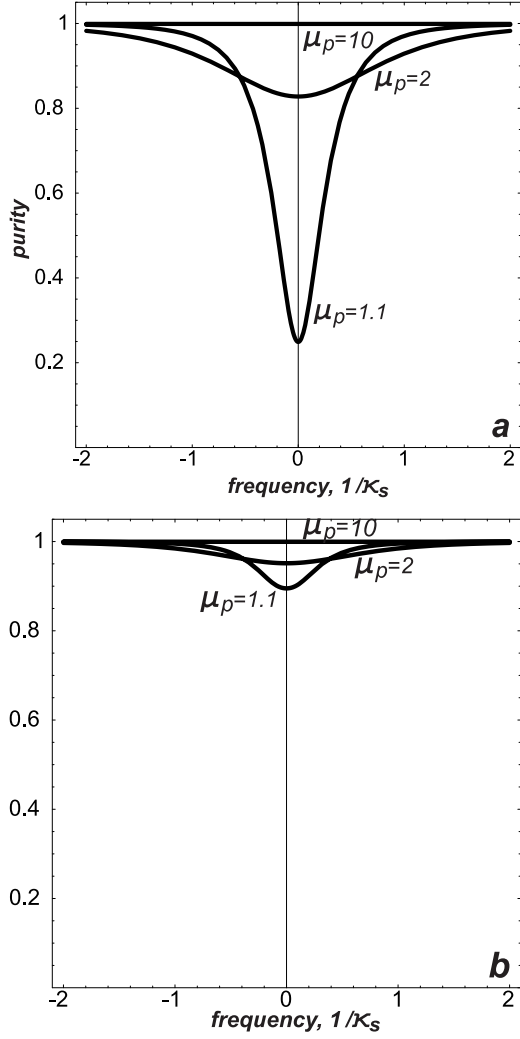
Fig. 1. Purity of the output field as a function of the dimensionless frequency for synchronizing field parameter (a) $\mu = 0.1$; (b) $\mu = 0.35$ and different excesses above threshold for symmetry phase locking. Both positive and negative frequency domains are plotted, however only the positive one correspond to physical values.

Figure 1a gives the spectral purity of the three-mode OPO for different values of the injection parameter ($\mu = 0.1$) and of the pump parameter ($\mu_p = 4.2$ and 1.1). One first observes that the spectral purity is equal to one outside the cavity bandwidth in all configurations, and also well above threshold at all frequencies. In contrast, the spectral purity at zero frequency turns out to be less than one near threshold ($\mu_p = 1.1$).

We are therefore led to the conclusion that, *close to threshold, the three-mode "single noise frequency state" describing the OPO is mixed at low noise frequencies.* We have taken into account in a quantum way all the interacting modes, and that all the intracavity modes are transmitted onto the detectors without any losses, so that the total input-output evolution matrix is unitary. But we must recall that we discuss here the purity not of the complete system, but the purity of some of its spectral com-
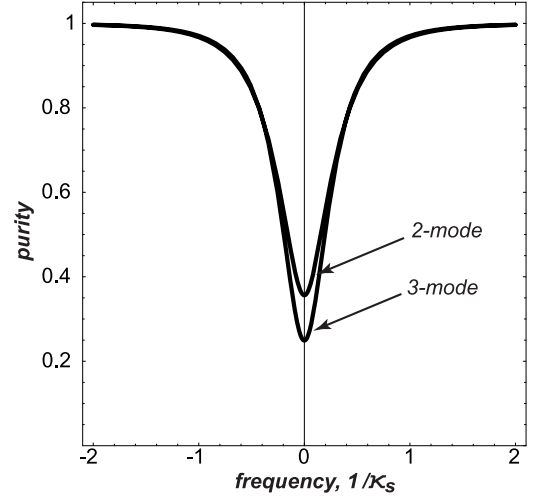

Fig. 2. Partial purity of the two-mode OPO compared to the system purity (including the pump mode). One note that the partial purity is always larger the total one.

ponents. Obviously, the intracavity parametric interaction does not change the purity of the system as a whole. However it leads to a redistribution of fluctuations and correlations between the spectral components of the pump, signal and idler modes, and as a result the purity of each spectral component does not generally survive. Such a redistribution is maximum when the coupling is maximum, i.e. at zero noise frequencies, and vanishes outside the cavity bandwidth. Figure 1b shows the spectral purity for larger injection. One can see that in this case the purity of the system is determined by the statistics of the injected field and becomes larger.

The non-degenerate OPO is often considered as a two-mode quantum system, for which the pump can be treated as a classical quantity. In this point of view, it is described by a two-mode quantum state, characterized by a density matrix which is the partial trace of $\hat{\rho}_\omega$ over the pump mode. Its spectral purity is related to the determinant of the two-mode covariance matrix. Such a "partial purity" is displayed in Figure 2. One observes that it is always larger than the purity of the total system. This can seem surprising that subsystem purity is larger that total purity, but it depends on the presence or absence of quantum correlations between parts of the system. For instance, when one only considers the differential mode, by tracing over the pump and sum modes, one finds that its partial purity is 1, and therefore that it is in a pure state. It is thereby possible to extract from our initial mixed state a pure subsystem by adequately eliminating the modes responsible for the "impurity" of the total state.

3.4 Spectral purity for an asymmetrical injection

In the previous subsection we have discussed the case of the symmetrical injection of the OPO when both the idler and signal modes with equal spectral widths $\kappa_i = \kappa_s$ are excited equally by the external fields with amplitudes

$\sqrt{N_i^{in}} = \sqrt{N_s^{in}}$. Here we want to briefly discuss the role of the asymmetry in the injection. The corresponding derivations can be found in the appendix of this paper, where the physical situation with $N_i^{in} = N^{in}$ and $N_s^{in} = 0$ is considered. In this case, the mean output signal and idler fields are no longer equal. In order to retrieve some symmetry we further require that $\kappa_s = \kappa_i(1 - \mu)$, so that the equality $N_i = N_s \equiv N$ survives for the stationary situation. Nevertheless the intracavity variances turn out to be different and essentially differently dependent on μ than under the symmetrical phase locking (see Appendix A). Besides, the relation between the variances of the output beam and the intracavity variances turns out to be more complicated for the $\pm$-variances:

$$2\langle\delta\hat{X}_{\pm}^2\rangle_\omega = 1 + \frac{\kappa_i}{8N}[(\varepsilon_{\pm}^2)_\omega(1 + \sqrt{1 - \mu})^2$$
$$+ (\varepsilon_{\mp}^2)_\omega(1 - \sqrt{1 - \mu})^2], \tag{73}$$
$$2\langle\delta\hat{Y}_{\pm}^2\rangle_\omega = 1 + \kappa_i N/2[(\varphi_{\pm}^2)_\omega(1 + \sqrt{1 - \mu})^2$$
$$+ (\varphi_{\mp}^2)_\omega(1 - \sqrt{1 - \mu})^2]. \tag{74}$$

Figure 3a shows the spectral dependence of the spectral purity in the asymmetrical case for a small injection field parameter $\mu = 0.1$. One can see first that outside the spectral band κ the purity is close to 1, as in the symmetrical case. The band where purity $\Pi_\omega \neq 1$, depends on the distance to threshold and decreases when the pump power increases. Well above threshold ($\mu_p > 2$) it depends on μ, and decreases when the injecting field decreases (see Fig. 3b).

However it is important to stress here that for the zero frequency we can never neglect the influence of the injection field even for very small values μ. In the limit of small μ and $\mu_p > 5$ we get $\Pi_{\omega=0} = 1/2$.

3.5 Multi-frequency squeezing and entanglement in presence of small injection

We can use formulas (63)–(70) to investigate the squeezing and entanglement in the OPO in presence of small injection. Even if this is not new in substance, we prove here that the presence of a small injected field do not destroy the non-classical features of the output fields.

Let us go back to the symmetrical case. Because

$$4\langle\delta\hat{X}_{i,s}^2\rangle_\omega = \langle\delta\hat{X}_+^2\rangle_\omega + \langle\delta\hat{X}_-^2\rangle_\omega \tag{75}$$

one has

$$4\langle\delta\hat{X}_{i,s}^2\rangle_\omega = 1 + \frac{1}{2}\frac{\kappa^2}{\kappa^2(\mu_p - 1 + \mu/4)^2 + \omega^2} - \frac{1}{2}\frac{\kappa^2}{\kappa^2 + \omega^2}. \tag{76}$$

Strongly above threshold $\mu_p \gg 1$ the second term turns out to be negligible and the maximum squeezing is reached in the idler and signal waves

$$4\langle\delta\hat{X}_{i,s}^2\rangle_{\omega=0} = \frac{1}{2}. \tag{77}$$

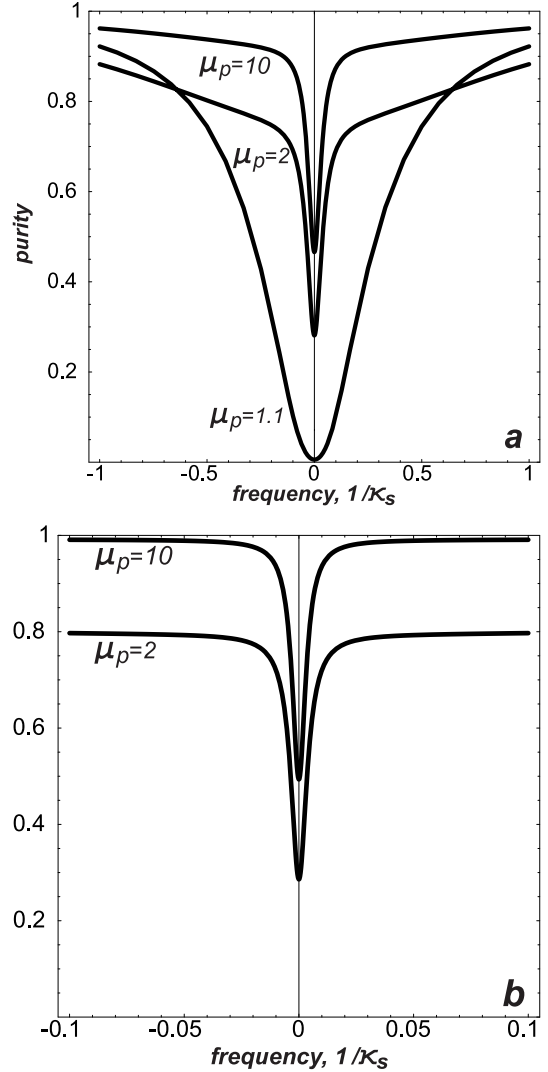We can expect squeezing for $\mu_p > 2$.

Fig. 3. Purity of the output field in dependence on dimensionless frequency for synchronizing field parameter (a) $\mu = 0.1$; (b) $\mu = 0.01$ and different excesses above threshold for asymmetry phase locking.

In contrast with the amplitude squeezing the phase squeezing is found only in the pump mode:

$$4\langle\delta\hat{Y}_p^2\rangle_\omega = 1 - \frac{2\kappa^2(\mu_p - 1)}{\kappa^2\mu_p^2 + \omega^2}. \tag{78}$$

On the zero frequency

$$4\langle\delta\hat{Y}_p^2\rangle_{\omega=0} = \frac{(\mu_p - 1)^2 + 1}{\mu_p^2}. \tag{79}$$

It is not difficult to see that maximum squeezing $4\langle\delta\hat{Y}_p^2\rangle_{\omega=0} = 1/2$ is reached as $\mu_p = 2$.

In order to evaluate an entanglement of the idler and signal waves, we use the Duan criterium

$$2\langle\delta\hat{X}_-^2\rangle, \ 2\langle\delta\hat{Y}_+^2\rangle < 1. \tag{80}$$

Taking into account (63) and (66) one can find that this criterium is carried out independently of pump. Even if, strongly above threshold ($\mu_p \gg 1$) $2\langle \delta \hat{Y}^2_{+\omega=0} \rangle \to 1$.

4 Stationary Glauber quasi-probability distribution

In the previous section we considered the properties of the ND-OPO from the point of view of its spectral components, by solving non stationary equations in Fourier space. Stationary master equations also contain important pieces of information about the quantum system. We devote this paragraph to the determination of the Glauber quasi-probability distribution of the present three-mode system in the limit of small amplitude and phase fluctuations which is relevant for the case of the injected ND-OPO. It is obtained by putting all time derivatives to zero in equations (19)–(22). We will discuss in the two following subsections the solutions of the amplitude and phase equations respectively.

4.1 Amplitude quasi-probability distribution

The stationary amplitude quasi-probability $P(\varepsilon_p, \varepsilon_i, \varepsilon_s)$ can be factorized in the form

$$P(\varepsilon_p, \varepsilon_i, \varepsilon_s) = P(\varepsilon_p, \varepsilon_+)P(\varepsilon_-), \qquad (81)$$

where each of the factors obeys its own equations:

$$\left(\kappa(1-\mu/2)\frac{\partial}{\partial \varepsilon_-}\varepsilon_- - \kappa N(1-\mu)\frac{\partial^2}{\partial \varepsilon_-^2} \right) P(\varepsilon_-) = 0, \tag{82}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial \varepsilon_p}(\kappa_p\varepsilon_p + \kappa(1-\mu)\varepsilon_+) + \frac{\partial}{\partial \varepsilon_+}(\kappa\mu/2\,\varepsilon_+ \right.$$
$$\left. - \kappa_p(\mu_p - 1)\varepsilon_p) + \kappa N(1-\mu)\frac{\partial^2}{\partial \varepsilon_+^2} \right) P(\varepsilon_p, \varepsilon_+) = 0. \tag{83}$$

Let us first consider the second equation. The solution has the Gaussian form:

$$P(\varepsilon_p, \varepsilon_+) = \frac{1}{2\pi\sqrt{D_p D_+}} \exp\left(-\frac{(\varepsilon_p - a\varepsilon_+)^2}{2D_p} - \frac{\varepsilon_+^2}{2D_+} \right). \tag{84}$$

The unknown parameters D_p, D_+ and a are coupled with the variances by means of the following relations:

$$D_+ = \overline{\varepsilon_+^2}, \qquad D_p = \overline{\varepsilon_p^2} - \frac{\overline{\varepsilon_+\varepsilon_p}^2}{\overline{\varepsilon_+^2}}, \qquad a = \frac{\overline{\varepsilon_+\varepsilon_p}}{\overline{\varepsilon_+^2}}. \tag{85}$$

As equation (83) provides us with a possibility to find the variances in the explicit form, then taking into account the previous inequalities we can calculate the parameters of the distribution D_p, D_+ and a. From equation (83) one

can then obtain the algebraic system of equation for the variances in the form:

$$-\kappa_p \overline{\varepsilon_p^2} - \kappa(1-\mu)\overline{\varepsilon_p\varepsilon_+} = 0, \tag{86}$$

$$-\frac{1}{2}(\kappa_p + \kappa\mu)\overline{\varepsilon_p\varepsilon_+} - \frac{1}{2}\kappa(1-\mu)\overline{\varepsilon_+^2} + \kappa_p(\mu_p-1)\overline{\varepsilon_p^2} = 0, \tag{87}$$

$$-\kappa\mu\overline{\varepsilon_+^2} + 2\kappa_p(\mu_p-1)\overline{\varepsilon_p\varepsilon_+} + 2\kappa N(1-\mu) = 0, \tag{88}$$

Solving this system, one can get the variances:

$$\overline{\varepsilon_+^2} = N\frac{1}{\mu/2 + \mu_p - 1}\left(1 + \frac{2\kappa}{\kappa_p + \kappa\mu}(\mu_p-1) \right), \tag{89}$$

$$\overline{\varepsilon_+\varepsilon_p} = -N_p\frac{1}{\mu/2 + \mu_p - 1}\frac{\kappa_p}{\kappa_p + \kappa\mu}(\mu_p-1), \tag{90}$$

$$\overline{\varepsilon_p^2} = N_p\frac{1}{\mu/2 + \mu_p - 1}\frac{\kappa}{\kappa_p + \kappa\mu}(\mu_p-1). \tag{91}$$

In the limit $\kappa, \kappa(\mu_p-1) \ll \kappa_p$ and $\mu \ll 1$ these variances read

$$\overline{\varepsilon_+^2} = N\frac{1}{\mu/2 + \mu_p - 1},$$
$$\overline{\varepsilon_p^2} = N_p\frac{\mu_p-1}{\mu/2 + \mu_p - 1}\frac{\kappa}{\kappa_p},$$
$$\overline{\varepsilon_+\varepsilon_p} = -N_p\frac{\mu_p-1}{\mu/2 + \mu_p - 1}. \tag{92}$$

The knowledge of the variances provides us with a possibility to find the parameters of the Gaussian quasi-probability:

$$D_+ = N\frac{1}{\mu/2 + \mu_p - 1},$$
$$D_p = N_p\frac{\mu_p-1}{\mu/2 + \mu_p - 1}\frac{\mu\kappa}{\kappa_p},$$
$$a = -\frac{\kappa}{\kappa_p}. \tag{93}$$

Let us discuss now the quasi-probability $P(\varepsilon_-)$ that is a solution of equation (82). This equation has a normalized solution only in the form of the distribution. This is connected with the minus sign in front of the second derivative with respect to ε_-. On the one hand this minus sign informs us about the quantum effects in the generation and on the other hand it makes impossible the derivation of the solution as a well-behaved <u>function</u>. The direct result of this situation is the variance $\overline{\varepsilon_-^2}$ turns out to be negative

$$\overline{\varepsilon_-^2} = -N \tag{94}$$

although the following relation

$$\overline{\varepsilon_-^{2k}} = (2k-1)!!\,\overline{\varepsilon_-^2}^k, \tag{95}$$

typical for the Gaussian distribution, survives. Knowing the variances allows us to derive the P-function as the formal series [17,18]:

$$P(\varepsilon_-) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{N}{2} \right)^k \frac{d^{2k}}{d\varepsilon_-^{2k}}\delta(\varepsilon_-). \tag{96}$$

It is also possible to use another formal equivalent expression:

$$P(\varepsilon_-) = \exp\left(-\frac{N}{2}\frac{d^2}{d\varepsilon_-^2}\right)\delta(\varepsilon_-). \qquad (97)$$

From these expressions, one can derive Fano parameters for each of the intracavity modes:

$$F_{i,s} = \frac{1}{4}\frac{2-\mu_p}{\mu/2+\mu_p-1}+1, \qquad F_p = 1. \qquad (98)$$

4.2 Phase quasi-probability distribution

As for the amplitude, the quasi-probability distribution of the three phases can be factorized in the form:

$$P(\varphi_p, \varphi_i, \varphi_s) = P(\varphi_p, \varphi_+)P(\varphi_-). \qquad (99)$$

The corresponding master equation read (19)–(22):

$$\frac{\partial P(\varphi_p, \varphi_+, t)}{\partial t} = \left(\frac{\kappa_p}{2}\frac{\partial}{\partial\varphi_p}(\varphi_p + (\mu_p-1)\varphi_+)\right.$$
$$+ \frac{\partial}{\partial\varphi_+}(\kappa(1-\mu/2)\varphi_+ - \kappa(1-\mu)\varphi_p)$$
$$\left. - \frac{\kappa}{4N}(1-\mu)\frac{\partial^2}{\partial\varphi_+^2}\right)P(\varphi_p, \varphi_+), \qquad (100)$$

$$\frac{\partial P(\varphi_-, t)}{\partial t} = \left(\frac{\mu\kappa}{2}\frac{\partial}{\partial\varphi_-}\varphi_- + \frac{\kappa}{4N}(1-\mu)\frac{\partial^2}{\partial\varphi_-^2}\right)P(\varphi_-). \qquad (101)$$

Again the minus sign in front of the last term means quantum phase features in the field. A formal solution of the equation can be presented in the form of the distribution:

$$P(\varphi_p, \varphi_+) = \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}}{m!\,n!}M_{mn}\frac{d^m}{d\varphi_p^m}\frac{d^n}{d\varphi_+^n}\delta(\varphi_p)\,\delta(\varphi_+),$$
$$M_{mn} = \overline{\varphi_p^m\,\varphi_+^n}. \qquad (102)$$

It is possible to demonstrate that the momenta M_{mn} are different from zero only as $m+n$ is the even number. Then, with help of equation (100), we are able to derive a recurrence relation connecting different non-zero even momenta with each other:

$$-(m\frac{\kappa_p}{2}+n\kappa)M_{mn} - m(\mu_p-1)\frac{\kappa_p}{2}M_{m-1n+1}$$
$$+ n\kappa M_{m+1n-1} - n(n-1)\frac{\kappa}{4N}M_{mn-2} = 0. \qquad (103)$$

In particular, as $\kappa, (\mu_p-1)\kappa \ll \kappa_p$

$$\overline{\varphi_p^2} = -\frac{1}{4N_p}\frac{\mu_p-1}{\mu_p}\frac{\kappa}{\kappa_p},$$
$$\overline{\varphi_+^2} = -\frac{1}{4N}\frac{1}{\mu_p},$$
$$\overline{\varphi_p\,\varphi_+} = \frac{1}{4N_p}\frac{1}{\mu_p}\frac{\kappa}{\kappa_p}. \qquad (104)$$

As for equation (101) it describes a stationary Gaussian process and its solution is well-known:

$$P(\varphi_-) = \frac{1}{\sqrt{\pi/(\mu N)}}\exp\left(-\frac{\varphi_-^2}{1/(\mu N)}\right). \qquad (105)$$

4.3 Stationary output quasi-probability distribution and purity

In the previous section, the output beams of the ND-OPO were presented as a set of mutually coupled field oscillators with different frequencies. In contrast, the P-function that we have just derived describes the system in its stationary state, but only for the three intracavity oscillators. Unfortunately, to the best of our knowledge, there is no simple relation between the intracavity P-function and some kind of quasi-probability distribution describing the system of the three OPO-modes when they have escaped the cavity. However it is possible to maintain a single-oscillator description of the three modes outside the cavity in a very specific case, as it has been explained by one of us reference [21]. The distribution function then only concerns output fields contained in "thin layers" of the propagation axis. The thickness of each of these layers must be much bigger than the wavelength, but much less than the cavity correlation length $l_0 = c\tau$, where τ the correlation time of the intracavity field. Then it is formally possible to introduce photon operators $\hat{a}_m$ and $\hat{a}_m^\dagger$ acting in mth layer such as $[\hat{a}_m, \hat{a}_n^\dagger] = \delta_{mn}$. This means that the field outside the cavity can be presented as a set of spatially located oscillators and that the propagation of the light in free space can be presented as a transfer of an excitation from one of the oscillators to the nearest one along the beam. These observables correspond to measurements performed on the output beams that integrate the photocurrent fluctuations over time scales much smaller than τ. One can then define a quasi-probability distribution P_{out} for such output oscillators, which is related to the already calculated intracavity P-function by means of

$$P_{out}(\alpha_{p,out}, \alpha_{i,out}, \alpha_{s,out}) =$$
$$\frac{1}{T_pT_iT_s}P\left(\frac{\alpha_{p,out}}{\sqrt{T_p}}, \frac{\alpha_{i,out}}{\sqrt{T_i}}, \frac{\alpha_{s,out}}{\sqrt{T_s}}\right) \qquad (106)$$

$\sqrt{T_i} = \sqrt{T_s} \equiv \sqrt{T}$ being the amplitude transmission coefficients of the coupling mirror for the signal and idler modes; $\sqrt{T_p}$ the transmission coefficient for the pump mode; $\alpha_{p,out}, \alpha_{i,out}\alpha_{s,out}$ are the corresponding complex amplitudes, which are the eigenvalues of the annihilation photon operator.

Knowing P_{out} of the whole system outside the cavity, at least in the restricted meaning described in the previous paragraph, it is now possible to calculate the purity of the stationary state of this system. It is the product of four factors:

$$\Pi_{st} = \Pi_1 \times \Pi_2 \times \Pi_3 \times \Pi_4, \qquad (107)$$

where the amplitude factors are

$$\Pi_1 = \iint d\varepsilon_p d\varepsilon_+ \, P_{out}(\varepsilon_p, \varepsilon_+) \int d\varepsilon_p' d\varepsilon_+' \, P_{out}(\varepsilon_p', \varepsilon_+')$$

$$\times \exp\left(-\frac{(\varepsilon_p - \varepsilon_p')^2}{4N_p T_p}\right) \exp\left(-\frac{(\varepsilon_+ - \varepsilon_+')^2}{8NT}\right), \quad (108)$$

$$\Pi_2 = \int d\varepsilon_- \, P_{out}(\varepsilon_-) \int d\varepsilon_-' \, P_{out}(\varepsilon_-')$$

$$\times \exp\left(-\frac{(\varepsilon_- - \varepsilon_-')^2}{8NT}\right), \quad (109)$$

and the phase ones are:

$$\Pi_3 = \iint d\varphi_p d\varphi_+ P_{out}(\varphi_p, \varphi_+) \iint d\varphi_+' d\varphi_p' P_{out}(\varphi_+', \varphi_p')$$

$$\times \exp\left(-\frac{(\varphi_p - \varphi_p')^2}{1/(N_p T_p)}\right) \exp\left(-\frac{(\varphi_+ - \varphi_+')^2}{2/(NT)}\right), \quad (110)$$

$$\Pi_4 = \int d\varphi_- \, P_{out}(\varphi_-) \int d\varphi_-' \, P_{out}(\varphi_-')$$

$$\times \exp\left(-\frac{(\varphi_- - \varphi_-')^2}{2/(NT)}\right). \quad (111)$$

Substituting equations (84), (96), (102), and (105) and considering the limit $\kappa, \ (\mu_p - 1)\kappa \ll \kappa_p$ we get

$$\Pi_1 = \frac{1}{(1 + \nu \, T/\mu)^{1/2}},$$
$$\Pi_2 = \Pi_3 = 1,$$
$$\Pi_4 = \frac{1}{(1 + T/\mu)^{1/2}}, \quad (112)$$

where

$$\nu = \frac{\mu(\mu_p - 1/2)}{\mu/2 + \mu_p - 1}. \quad (113)$$

If the output mirror of the cavity is highly reflecting, so that that $T \ll \mu$, then

$$\Pi_{st} = 1. \quad (114)$$

A similar calculation performed with the intracavity P-function with the same parameters gives $\Pi_1 = \Pi_2 = \Pi_3 = 1$, and $\Pi_4 = \sqrt{\mu}$, so that the purity of the intracavity state, $\Pi = \mu^{1/2}$ is much smaller than 1.

Field fluctuations integrated over short time intervals depend only on the high frequency fluctuations of the output fields. The property that $\Pi_{st} = 1$ is therefore certainly connected to the already noticed feature that $\Pi_\omega = 1$ for noise frequencies larger than the cavity bandwidth.

5 Stationary photon number probability

Another important way of characterizing the quantum state produced by the TROPO is to determine the full photon number probability distribution in each of the

three beams exiting the system. This quantity is important to know, for example when one wants to determine the quantum state which is produced by a conditional measurement performed on the intensity of one of the output beams. This is what we will do in this section. More precisely, we will determine the probability for n photons to cross the cross section of the output beam during a given time τ. Let us stress that this function is essentially different than the already calculated stationary probability for the photon number because it includes the integration time τ.

Let us first calculate the photon number probability inside the cavity. The joint probability $C_{in}(n_i, n_s)$ to find n_i photons in the idler intracavity mode and simultaneously n_s photons in the signal intracavity mode is defined as

$$C_{in}(n_i, n_s) = \sum_{n_p = 0, 1, \ldots} \langle n_p n_i n_s | \hat{\rho} | n_p n_s n_i \rangle, \quad (115)$$

where $\hat{\rho}$ is the stationary three-mode intracavity density matrix. By using the Glauber diagonal representation (7) in (78), we have

$$C_{in}(n_i, n_s) = \frac{1}{4} \int_0^\infty \int_0^\infty du_i \, du_s P_{red.}(u_i, u_s)$$

$$\times e^{-u_i} \frac{u_i^{n_i}}{n_i!} \, e^{-u_s} \frac{u_s^{n_s}}{n_s!}, \quad (116)$$

where $P_{red.}$ is a "reduced" Glauber photon quasi-probability given by

$$P_{red.}(u_i, u_s) = \iint d^2\alpha_p \iint d\varphi_i d\varphi_s P(\alpha_p, \alpha_i, \alpha_s), \quad (117)$$

where $\alpha_l = \sqrt{u_l} \, \exp(i\varphi_l) \ (l = s, i)$. From the previous section

$$P_{red.}(u_i, u_s) = P(\varepsilon_-) \int d\varepsilon_p P(\varepsilon_p, \varepsilon_+)$$

$$= \frac{1}{\sqrt{2\pi D_+}} \exp\left(-\frac{\varepsilon_+^2}{2D_+}\right) \exp\left(-\frac{N}{2} \frac{d^2}{d\varepsilon_-^2}\right) \delta(\varepsilon_-). \quad (118)$$

Substituting this into (116) one can get after the corresponding integrating one can obtain that

$$C_{in}(n_i, n_s) = 1/\sqrt{2\pi\lambda N} \, \exp\left(-\frac{(n_+ - 2N)^2}{2\lambda N}\right)$$

$$\times 1/\sqrt{2\pi N} \exp\left(-\frac{n_-^2}{2N}\right), \quad (119)$$

where

$$\lambda = 2 + \frac{1}{\mu/4 + \mu_p - 1} \quad (120)$$

$$(\kappa, \ (\mu_p - 1)\kappa \ll \kappa_p).$$

This function describes the probabilities to find n_i photons in the idler mode and n_s photons in the signal mode inside

the cavity in the stationary regime. Although knowing the intracavity probability is not enough for determining the probability for the output travelling fields, it will helps us to guess the general form of the wished function. We will assume that the outside probability for output fields has the same Gaussian form as the probability for the intra-cavity fields, but with its own parameters depending, in particular, on observation time τ. We will therefore write it as:

$$C_{out}(n_i, n_s) = 1/\sqrt{2\pi d_+} \, \exp\left(-\frac{(n_+ - 2N_{out})^2}{2d_+}\right)$$
$$\times 1/\sqrt{2d_-} \, \exp\left(-\frac{n_-^2}{2d_-}\right) \qquad (121)$$

where the parameters N_{out}, $d_{\pm}$ depend on τ. The way to calculate the parameters will be demonstrated on the simpler photon probability of a single mode.

By integrating (119) over one of the variances ε_i or ε_s we obtain the single-mode photon number probability for both the idler and signal modes:

$$C_{in}(n_l) = \frac{1}{\sqrt{2\pi N F_{in}}} \, \exp\left(-\frac{(n_l - N)^2}{2N F_{in}}\right), \quad (122)$$
$$F_{in} = \frac{1}{4}(1 + \lambda), \qquad l = i, s,$$

and correspondingly for the output beam:

$$C_{out}(n_l) = \frac{1}{\sqrt{2\pi F_{out}}} \, \exp\left(-\frac{(n_l - N_{out})^2}{2N F_{out}}\right). \quad (123)$$

In the last probability n_l are eigen-numbers of the operator

$$\hat{n}_l(t) = \int\limits_{t-\tau/2}^{t+\tau/2} \hat{A}_l^\dagger(t') \, \hat{A}_l(t') \, dt', \qquad (124)$$

where the operators $\hat{A}_l$ and $\hat{A}_l^\dagger$ describe light propagating in free space (see (46)). Then

$$N_{out} = \int\limits_{t-\tau/2}^{t+\tau/2} \langle \hat{A}_l^\dagger(t') \, \hat{A}_l(t') \rangle \, dt' = \kappa\tau N. \quad (125)$$

In order to determine the Fano-factor F_{out}, we have to calculate the variance

$$\langle \hat{n}_l^2 \rangle - \langle \hat{n}_l \rangle^2 = N_{out} F_{out}. \qquad (126)$$

For the stationary flux

$$N_{out} F_{out} =$$
$$\int\limits_{-\tau/2}^{\tau/2} \int\limits_{-\tau/2}^{\tau/2} dt_1 dt_2 \langle \hat{A}_l^\dagger(t_1) \hat{A}_l(t_1) \hat{A}_l^\dagger(t_2) \hat{A}_l(t_2) \rangle - N_{out}^2. \quad (127)$$

Following to standard procedure, one obtains

$$F_{out} = 1 + \kappa/N \int d\omega \, (\varepsilon_l^2)_\omega \, \delta_\tau(\omega), \qquad (128)$$

where the integral contains a product of two functions. One of them is

$$\kappa/N \, (\varepsilon_l^2)_\omega = \frac{1}{2}\left(\frac{\kappa^2}{(\mu/2 + \mu_p - 1)^2\kappa^2 + \omega^2} - \frac{\kappa^2}{\kappa^2 + \omega^2}\right). \qquad (129)$$

It is the spectral density of the amplitude noise in the selected l-wave that can be obtained from (35) and (37). The other is

$$\delta_\tau(\omega) = \frac{1}{\pi} \, \frac{\sin^2 \omega\tau/2}{\tau\omega^2/2}. \qquad (130)$$

For long enough integration times τ, $\delta_\tau(\omega)$ gets close to a δ-function when $\tau \to \infty$, so that F_{out} is given by

$$F_{out} = \frac{1}{2} \, \frac{\mu_p}{\mu/2 + \mu_p - 1}. \qquad (131)$$

Near threshold $F_{out} = \mu^{-1} \gg 1$, so that the photon statistics of the signal or idler field turns out to be super-Poissonian. In contrast, strongly above threshold $F_{out} = 1/2$, and the photon statistics of the signal or idler field is now sub-Poissonian, as it has been already noticed and experimentally checked (Kasai).

The joint signal-idler photon probability distribution can be derived exactly in the same way, knowing the two-mode variances $d_{\pm}$ (119):

$$d_{\pm} = 2N_{out}\left(1 + \kappa/(2N) \int d\omega \, (\varepsilon_{\pm}^2)_\omega \, \delta_\tau(\omega)\right). \quad (132)$$

As $(\mu_p - 1)\kappa, \kappa \ll \kappa_p$

$$\kappa/(2N) \, (\varepsilon_+^2)_\omega = \frac{\kappa^2}{(\mu/4 + \mu_p - 1)^2\kappa^2 + \omega^2},$$
$$\kappa/(2N) \, (\varepsilon_-^2)_\omega = -\frac{\kappa^2}{\kappa^2 + \omega^2}. \qquad (133)$$

Inserting these expressions to formula (132), one obtains the full joint photon number probability distribution $C_{out}(n_i, n_s)$ of the output signal and idler beams for any observation time τ.

If we choose the short observation time τ such as $\kappa\tau$, $\kappa\tau(\mu/4 + \mu_p - 1) \ll 1$ the variances read:

$$d_{\pm} = 2N_{out}. \qquad (134)$$

This means that under the observation for only short time the photon statistics turn out to be Poissonian.

On the other hand, for $\kappa\tau$, $\kappa\tau(\mu/4 + \mu_p - 1) \gg 1$,

$$d_+ = \mu_p/(\mu/2 + \mu_p - 1), \quad d_- = 0. \qquad (135)$$

We have a possibility to derive the wished photon number probability in form:

$$C_{out}(n_i, n_s) = \frac{1}{\sqrt{2\pi d_+}} \exp\left(-\frac{(2n_i + -2N_{out})^2}{2d_+}\right) \delta(n_i - n_s).$$
(136)

So if we count photons during long time intervals, the photon numbers in both waves turn out to be the same. This the form of the joint probability distribution that was guessed in [23], and used to calculate the state of the signal mode produced by a conditional measurement performed on the idler mode.

6 Conclusion

One of our aims here was to investigate the quantum state purity of the TROPO radiation. There is a widely-distributed opinion that the purity of the output emission state must be equal to one. The reason of that is connected with a representation about the OPO as system that, in the unitary process, converts the pure state of the input light (the pump wave is in the coherent state and the other modes are in the vacuum states or the pump, idler, signal waves are in the coherent states and again the other modes are in the vacuum states) into the same pure state of the output light. This conclusion absolutely correct for the output field as whole nevertheless does not concern the states for the selected frequencies, and just it is an object of the investigation in experiments and theories. For example, the covariance matrix is constructed as a combination of the quadrature variances for the selected frequency. The purity for this oscillator turns out to be uncertain and, in principle, can accept any meanings.

We have considered the purity and been convinced that, strictly speaking, the purity for the state with selected frequency is not one (especially near zero frequencies). However for the symmetrical synchronization the purity turns out to be close to one, provided the pump power is strongly above threshold. At the same time under the asymmetrical synchronization the purity becomes $1/2$ even if $\mu_p \gg 1$. So we can conclude that, generally speaking, the purity for the oscillator on the selected frequency is not one and essentially depends on the power of the synchronizing field and the asymmetry of the synchronization too.

We needed to introduce the synchronization into our the OPO system to depress the diffusion of the differential phase between signal and idler waves. Usually, it is suggested that the synchronization especially by the weak external field does not insert any essential distortions into a statistical pattern of the TROPO except a phase diffusion depression. However, strictly speaking, this is quite not obvious and one of our aims here was to investigate just this side of the OPO generation. As was mentioned already the purity can be essentially dependent on the synchronization.

Besides although we introduce the synchronization to depress the phase diffusion, nevertheless not only the phase fluctuations are stabilized under an influence of the external field but the amplitude ones too. We have found that this phenomenon turns out to be essential near threshold as $\mu_p - 1 < \mu$. One can see that all amplitude variances on the zero frequency are proportional to $1/\mu^2$ (not to infinity as under the phase diffusion). This apiaries as well as in the spectral and stationary variances.

Appendix A: Case of an asymmetrical injection

A.1 Master equations

In the main part of the article, we discussed in detail how the phase locking phenomenon produced by a symmetrical injection on the idler and signal modes by an external coherent light acts on the statistical properties of the OPO radiation. In this appendix we consider the case of an asymmetrical injection, more precisely when the idler mode is injected by a coherent field with amplitude $\sqrt{N_i^{in}} = \sqrt{N^{in}}$ and the signal mode by the vacuum state $\sqrt{N_s^{in}} = 0$. Let us first rewrite the master equation in the form:

$$\frac{\partial P(\alpha_p, \alpha_i, \alpha_s, t)}{\partial t} = \frac{\kappa_i}{2}\frac{\partial}{\partial \alpha_s}\alpha_s P + \frac{\kappa_s}{2}\frac{\partial}{\partial \alpha_i}(\alpha_i - \sqrt{N^{in}})P$$
$$+ \frac{\kappa_p}{2}\frac{\partial}{\partial \alpha_p}(\alpha_p - \sqrt{N_p^{in}})P + g\left(\alpha_i\alpha_s\frac{\partial P}{\partial \alpha_p} - \alpha_p\alpha_s^*\frac{\partial P}{\partial \alpha_i}\right.$$
$$\left. -\alpha_p\alpha_i^*\frac{\partial P}{\partial \alpha_s}\right) + g\alpha_p\frac{\partial^2 P}{\partial \alpha_i \partial \alpha_s} + c.c.$$
(137)

The corresponding classical equations read

$$\dot{\alpha}_p = -\frac{\kappa_p}{2}\left(\alpha_p - \sqrt{N_p^{in}}\right) - g\alpha_i\alpha_s \tag{138}$$

$$\dot{\alpha}_i = -\frac{\kappa_i}{2}\left(\alpha_i - \sqrt{N^{in}}\right) + g\alpha_p\alpha_s^* \tag{139}$$

$$\dot{\alpha}_s = -\frac{\kappa_s}{2}\,\alpha_s + g\alpha_p\alpha_i^*. \tag{140}$$

Putting

$$\kappa_i = \kappa_s(1 - \mu) \tag{141}$$

one can obtain the stationary solutions of the classical equations in the form:

$$\alpha_p = \sqrt{N_p}, \qquad \alpha_{i,s} = \sqrt{N}, \tag{142}$$

where

$$N_p = \frac{\kappa_s^2}{4g^2}, \qquad N = (\mu_p - 1)\frac{\kappa_s\kappa_p}{4g^2}, \tag{143}$$

and

$$\kappa_s N = (1 - \mu)\kappa_i N = (\mu_p - 1)\kappa_p N_p. \qquad (144)$$

In the limit of small amplitude and phase fluctuations it is possible to factorize the solutions in its phase and amplitude factors:

$$P(\alpha_p, \alpha_i, \alpha_s, t) = P(\varepsilon_p, \varepsilon_i, \varepsilon_s, t)P(\varphi_p, \varphi_i, \varphi_s, t). \quad (145)$$

We now introduce to usual sum and difference notations

$$\varepsilon_\pm = \varepsilon_i \pm \varepsilon_s, \qquad \varphi_\pm = \varphi_i \pm \varphi_s \qquad (146)$$

and convert equation (137) into two equations:

$$\frac{\partial P(\varepsilon_p, \varepsilon_+, \varepsilon_-, t)}{\partial t} = \left(\frac{1}{2}\frac{\partial}{\partial \varepsilon_p}(\kappa_p \varepsilon_p + \kappa_s \varepsilon_+)\right.$$
$$+ \frac{\partial}{\partial \varepsilon_+}\left(\frac{\mu \kappa_i}{4}\varepsilon_+ - \frac{(\mu_p - 1)\kappa_p}{\sqrt{1-\mu}}\varepsilon_p\right) + \kappa_s N \frac{\partial^2}{\partial \varepsilon_+^2}$$
$$\left. + \kappa_i\left(1 - \frac{3\mu}{4}\right)\frac{\partial}{\partial \varepsilon_-}\varepsilon_- - \kappa_s N \frac{\partial^2}{\partial \varepsilon_-^2}\right) P(\varepsilon_p, \varepsilon_+, \varepsilon_-, t)$$
$$+ \mu\left(\frac{\partial P}{\partial t}\right)_1, \qquad (147)$$

$$\frac{\partial P(\varphi_p, \varphi_+, \varphi_-, t)}{\partial t} = \left(\frac{\kappa_p}{2}\frac{\partial}{\partial \varphi_p}(\varphi_p + \frac{(\mu_p - 1)}{\sqrt{1-\mu}}\ \varphi_+)\right.$$
$$+ \frac{\partial}{\partial \varphi_+}\left(\kappa_i\left(1 - \frac{3\mu}{4}\right)\varphi_+ - \kappa_s \varphi_p\right)$$
$$- \frac{\kappa_s}{4N}\frac{\partial^2}{\partial \varphi_+^2} + \frac{\mu \kappa_i}{4}\frac{\partial}{\partial \varphi_-}\varphi_-$$
$$\left. + \frac{\kappa_s}{4N}\frac{\partial^2}{\partial \varphi_-^2}\right) P(\varphi_p, \varphi_+, \varphi_-, t) + \mu\left(\frac{\partial P}{\partial t}\right)_1. \qquad (148)$$

Here the selected terms on the right of the equations have the following form

$$\left(\frac{\partial P}{\partial t}\right)_1 = \frac{\kappa_i}{4}\left(\frac{\partial}{\partial \varepsilon_+}\varepsilon_- + \frac{\partial}{\partial \varepsilon_-}\varepsilon_+\right) P(\varepsilon_p, \varepsilon_+, \varepsilon_-, t),$$
$$(149)$$

$$\left(\frac{\partial P}{\partial t}\right)_1 = \frac{\kappa_i}{4}\left(\frac{\partial}{\partial \varphi_+}\varphi_- + \frac{\partial}{\partial \varphi_-}\varphi_+\right) P(\varphi_p, \varphi_+, \varphi_-, t).$$
$$(150)$$

One can see that their role is essential when the synchronizing parameter μ is big. Then some additional mixing the variables ε_l takes place. To avoid it we require that $\mu \ll 1$. A similar remark concerns the variables φ_l. In our limit we neglect these terms, so that the Glauber quasi-probability is factorized in the form

$$P(\alpha_p, \alpha_i, \alpha_s, t) = P(\varepsilon_p, \varepsilon_+, t)P(\varepsilon_-, t)P(\varphi_p, \varphi_+, t)P(\varphi_-, t)$$
$$(151)$$

and correspondingly the equations are decoupled as

$$\frac{\partial P(\varepsilon_p, \varepsilon_+, t)}{\partial t} = \left(\frac{1}{2}\frac{\partial}{\partial \varepsilon_p}(\kappa_p \varepsilon_p + \kappa_s \varepsilon_+)\right.$$
$$+ \frac{\partial}{\partial \varepsilon_+}\left(\frac{\mu \kappa_i}{4}\varepsilon_+ - \frac{(\mu_p - 1)\kappa_p}{\sqrt{1-\mu}}\varepsilon_p\right)$$
$$\left. + \kappa_s N \frac{\partial^2}{\partial \varepsilon_+^2}\right) P(\varepsilon_p, \varepsilon_+, t), \qquad (152)$$

$$\frac{\partial P(\varepsilon_-, t)}{\partial t} = \left(\kappa_i(1 - 3\mu/4)\frac{\partial}{\partial \varepsilon_-}\varepsilon_- - \kappa_s N \frac{\partial^2}{\partial \varepsilon_-^2}\right) P(\varepsilon_-, t),$$
$$(153)$$

$$\frac{\partial P(\varphi_p, \varphi_+, t)}{\partial t} = \left(\frac{\kappa_p}{2}\frac{\partial}{\partial \varphi_p}(\varphi_p + \frac{(\mu_p - 1)}{\sqrt{1-\mu}}\varphi_+)\right.$$
$$\left. + \frac{\partial}{\partial \varphi_+}(\kappa_i(1 - 3\mu/4)\varphi_+ - \kappa_s \varphi_p) - \frac{\kappa_s}{4N}\frac{\partial^2}{\partial \varphi_+^2}\right) P(\varphi_p, \varphi_+, t),$$
$$(154)$$

$$\frac{\partial P(\varphi_-, t)}{\partial t} = \left(\frac{\mu \kappa_i}{4}\frac{\partial}{\partial \varphi_-}\varphi_- + \frac{\kappa_s}{4N}\frac{\partial^2}{\partial \varphi_-^2}\right) P(\varphi_-, t).$$
$$(155)$$

In order to analyze the time dependent (spectral) correlation functions, we use the Langevin equations of the system, which are have the following form:

$$\dot{\varepsilon}_p = -\kappa_p/2\ \varepsilon_p - \kappa_s/2\ \varepsilon_+, \qquad (156)$$
$$\dot{\varepsilon}_+ = -\kappa_i \mu/4\ \varepsilon_+ + \kappa_p(\mu_p - 1)/\sqrt{1-\mu}\ \varepsilon_p + f_+(t), \qquad (157)$$
$$\dot{\varepsilon}_- = -\kappa_i(1 - 3\mu/4)\ \varepsilon_- + f_-(t), \qquad (158)$$
$$\dot{\varphi}_p = -\kappa_p/2\ \varphi_p - \kappa_p(\mu_p - 1)/(2\sqrt{1-\mu})\ \varphi_+, \qquad (159)$$
$$\dot{\varphi}_+ = -\kappa_i(1 - 3\mu/4)\ \varphi_+ + \kappa_s\ \varphi_p + g_+(t), \qquad (160)$$
$$\dot{\varphi}_- = -\kappa_i \mu/4\ \varphi_- + g_-(t), \qquad (161)$$

where the stochastic sources are determined by the pair correlation functions

$$\langle f_+(t)f_+(t')\rangle = 2\kappa_s N\ \delta(t - t'), \qquad (162)$$
$$\langle f_-(t)f_-(t')\rangle = -2\kappa_s N\ \delta(t - t'), \qquad (163)$$
$$\langle g_+(t)g_+(t')\rangle = -\kappa_s/(2N)\ \delta(t - t'), \qquad (164)$$
$$\langle g_-(t)g_-(t')\rangle = \kappa_s/(2N)\ \delta(t - t'). \qquad (165)$$

The solution of these equations is found in the Fourier transforms $\varepsilon_{\pm,\omega}, \varepsilon_{p,\omega}, \varphi_{\pm,\omega}, \varphi_{p,\omega}$. Their variances and correlation functions are given by:

$$(\varepsilon_+^2)_\omega = 2N\kappa_s(\kappa_p^2 + 4\omega^2)/\Lambda_\varepsilon, \qquad (166)$$
$$(\varepsilon_p^2)_\omega = 2N_p(\mu_p - 1)\kappa_i \kappa_s \kappa_p/\Lambda_\varepsilon, \qquad (167)$$
$$(\varepsilon_p \varepsilon_+)_\omega = -2N\kappa_s^2 \kappa_p/\Lambda_\varepsilon, \qquad (168)$$
$$(\varepsilon_-^2)_\omega = -2N\frac{\kappa_s}{\kappa_i^2(1 - 3\mu/4)^2 + \omega^2}, \qquad (169)$$

and

$$(\varphi_p^2)_\omega = -\frac{1}{2N_p\Lambda_\varphi}(\mu_p - 1)\kappa_i^2\kappa_p, \tag{170}$$

$$(\varphi_+^2)_\omega = -\frac{1}{2N\Lambda_\varphi}\kappa_s(\kappa_p^2 + 4\omega^2), \tag{171}$$

$$(\varphi_p\varphi_+)_\omega = \frac{1}{2N\Lambda_\varphi}\kappa_i\kappa_p^2(\mu_p - 1)\sqrt{1 - \mu}, \tag{172}$$

$$(\varphi_-^2)_\omega = \frac{1}{2N}\,\frac{\kappa_s}{(\kappa_i\mu/4)^2 + \omega^2}. \tag{173}$$

where

$$\Lambda_\varphi = \left[2\omega^2 - \kappa_i\kappa_p[1 - 3\mu/4 + (\mu_p - 1)\sqrt{1 - \mu}\,]\right]^2$$
$$+ \omega^2[\kappa_p + 2\kappa_i(1 - 3\mu/4)]^2. \tag{174}$$

$$\Lambda_\varepsilon = \left[2\omega^2 - \kappa_p\kappa_i[\mu/4 + (\mu_p - 1)\sqrt{1 - \mu}\,]\right]^2$$
$$+ \omega^2(\kappa_p + \kappa_i\mu/2)^2. \tag{175}$$

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